

# DIFFRACTION INTENSITIES OF A CLASS OF BINARY PISOT SUBSTITUTIONS VIA EXPONENTIAL SUMS

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**ABSTRACT.** This paper is concerned with the study of diffraction intensities of a relevant class of binary Pisot substitutions via exponential sums. Arithmetic properties of algebraic integers are used to give a new and constructive proof of the fact that there are no diffraction intensities outside the Fourier module of the underlying cut and project schemes. The results are then applied in the context of random substitutions.

## 1. INTRODUCTION

The aim of mathematical diffraction theory is to describe the structure of point configurations in space (which model crystals and quasicrystals) through the associated autocorrelation and diffraction measures. Bombieri and Taylor were among the first to raise the question which distributions of matter diffract, i.e. show sharp spots (or Bragg peaks) in their diffraction patterns; see [6]. There are two successful approaches to generate such structures. The first is by creating certain tilings by the method of inflation followed by decomposition using a finite set of proto-tiles. The second is by creating point sets through the method of cut and project sets, which are also called model sets. Either way, it is desirable to obtain explicit formulas for the associated diffraction measure; see [1, Thm. 9.4] and [1, Prop. 9.9]. However, some parts of the proof of [1, Prop. 9.9] are not constructive and require knowledge of abstract harmonic analysis, which is due to [13, Prop. 4.5.1].

The objective of this paper is to give a constructive and elementary proof of pure point diffraction for a certain class of binary Pisot substitutions via exponential sums. This will be done in Section 3, Theorem 7. The key ingredients are arithmetic properties of powers of algebraic integers  $\alpha$ , i.e. expressions of the form  $\{\xi\alpha^n\}$ ,  $\xi \in \mathbb{R}$ ; compare Lemmas 4 and 5. Here,  $\{x\}$  denotes the fractional part of  $x$ ,  $[x]$  denotes the integer part of  $x$  and

$$\|x\| := \min\{|x - m| \mid m \in \mathbb{Z}\}.$$

These were investigated by Dubickas in [8, 9]. In Section 4, the main result will be extended to random inflation tilings, i.e. the result states that there are no pure point diffraction intensities outside the Fourier module for a special class of stochastic substitutions. These one-dimensional tilings, which belong to a class of tilings first considered by Godrèche and Luck in [10], extend the study of conventional substitutions and introduce the notion of local mixtures of substitution rules on the basis of a fixed probability vector; see [15, 16] for further details.

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## 2. PRELIMINARIES

The purpose of this section is to summarise results from mathematical diffraction theory; see [1, Chs. 8 and 9] for general background. Let  $P$  be an infinite uniformly discrete point set in  $\mathbb{R}$ . Define the attributed *Dirac comb* by

$$\delta_P := \sum_{x \in P} \delta_x \quad \text{together with} \quad \widetilde{\delta_P} := \sum_{y \in P} \delta_{-y}$$

and study the properties of the family of measures  $\{\gamma_P^{(n)} \mid n > 0\}$  with

$$\gamma_P^{(n)} := \gamma_{\delta_P}^{(n)} := \frac{\delta_{P_n} * \widetilde{\delta_{P_n}}}{\text{vol}(B_n)}$$

and  $P_n := P \cap B_n(0)$ .

It is not clear that the sequence  $(\gamma_P^{(n)})_{n \in \mathbb{N}}$  converges. Each  $\gamma_P^{(n)}$  is well-defined (since  $\delta_{P_n}$  is a finite measure with compact support) and positive definite by construction. Every accumulation point of  $\{\gamma_P^{(n)} \mid n > 0\}$  in the vague topology is called an *autocorrelation measure* of  $\delta_P$ , and as such it is a positive definite measure by construction. If only one accumulation point exists, the autocorrelation measure

$$\gamma_P := \lim_{n \rightarrow \infty} \gamma_P^{(n)}$$

is well-defined and Fourier transformable. Its Fourier transform  $\widehat{\gamma_P}$  is called *diffraction measure*. In the case of a cut and project scheme (CPS), there are explicit formulas for the diffraction measure; for a detailed introduction of cut and project schemes, we refer the reader to [1, Ch. 7], [17] as well as [14].

**Theorem 1.** [1, Thm. 9.4] *Let  $\Lambda = \lambda(W)$  be a regular model set for the CPS  $(\mathbb{R}, H, \mathcal{L})$  with compact window  $W = \overline{W^\circ}$  and autocorrelation  $\gamma_\Lambda$ . The diffraction measure  $\widehat{\gamma_\Lambda}$  is a positive and positive definite, translation bounded measure. It is explicitly given by*

$$\widehat{\gamma_\Lambda} = \sum_{k \in L^\circ} I(k) \delta_k,$$

where the diffraction intensities are  $I(k) = |A(k)|^2$  with the amplitudes

$$A(k) = \frac{\text{dens}(\Lambda)}{\mu_H(W)} \widehat{1_W}(-k^\star)$$

and supporting set  $L^\circ = \pi_1(\mathcal{L}^\star)$ , and  $\mathcal{L}^\star$  is the dual lattice of  $\mathcal{L}$ . □

Furthermore, there is an alternative approach via exponential sums which is justified by [11, Thm. 3.4], because we deal with pure point measures.

**Proposition 2.** [1, Prop. 9.9] *Consider a regular model set  $\Lambda = \lambda(W)$  for the CPS  $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ , with compact window  $W = \overline{W^\circ}$  and Fourier module  $L^\circ = \pi_1(\mathcal{L}^\star) \subseteq \mathbb{R}$ . Then, one has*

$$\frac{1}{|B_N|} \sum_{x \in \Lambda_N} e^{-2\pi i k x} \xrightarrow{N \rightarrow \infty} \begin{cases} A(k), & k \in L^\circ, \\ 0, & \text{otherwise,} \end{cases}$$

where  $A(k)$  is the amplitude of Theorem 1 for the internal space  $H = \mathbb{R}$ . □

The proof of this result is constructive when  $k \in L^*$  or  $k \in \mathbb{Q}(L^*)$ , in the sense that it uses an explicit convergence argument via exponential sums. For  $k \notin \mathbb{Q}(L^*)$ , however, it uses an abstract argument from [13]. In what follows, we demonstrate an alternative approach via exponential sums, for a relevant class of Pisot substitutions.

### 3. INTENSITIES OF A CERTAIN CLASS OF PISOT SUBSTITUTIONS

Consider the binary alphabet  $\mathcal{A} = \{a, b\}$  with the dictionary  $\mathcal{A}_2^*$  and the substitution

$$\sigma : \mathcal{A}_2^* \rightarrow \mathcal{A}_2^*, \quad \sigma : \begin{cases} a \mapsto w(a, b) \\ b \mapsto a \end{cases},$$

where  $w(a, b)$  is a word in  $a$  and  $b$ , in which the letter  $a$  occurs  $p$  times and the letter  $b$  occurs  $q$  times with  $p, q \in \mathbb{N}$ ,  $p \geq q$ . The eigenvalues of the substitution matrix

$$M_\sigma = \begin{pmatrix} p & 1 \\ q & 0 \end{pmatrix}$$

are

$$\vartheta := \frac{p + \sqrt{p^2 + 4q}}{2} \quad \text{und} \quad \vartheta' := \frac{p - \sqrt{p^2 + 4q}}{2}.$$

The assumption  $q \leq p$  guaranties that  $\sigma$  is a *Pisot substitution*. A primitive substitution is called Pisot substitution if the Perron–Frobenius eigenvalue  $\lambda_{\text{PF}}$  is a Pisot–Vijayaraghavan (PV) number, i.e. an algebraic integer strictly greater than 1 whose conjugates lie inside the open unit disc,  $\{z \in \mathbb{C} \mid |z| < 1\}$ .

The left eigenvector  $v = (\vartheta, 1)$  gives rise to a *geometric* realisation of  $\sigma$  as a tiling of the real line by two types of intervals; see [1, p. 74]. To calculate the diffraction intensities of these tilings, we need some preparation.

Define  $w^{(0)} := b$  and  $w^{(n)} := \sigma^n(b)$  for  $n \in \mathbb{N}$ . This sequence of words  $(w^{(n)})_{n \in \mathbb{N}}$  satisfies a *concatenation rule*, i.e. if  $w(a, b) = w_0 \dots w_{|w(a, b)|-1}$ , we have

$$w^{(n)} = w^{(j_0)} \dots w^{(j_{|w(a, b)|-1})},$$

where  $j_i = n - 1$  if  $w_i = a$  and  $j_i = n - 2$  if  $w_i = b$  (in the case  $w(a, b) = ab$ , the Fibonacci chain, this is  $w^{(n)} = w^{(n-1)}w^{(n-2)}$ ). Moreover, we consider a linear recursion

$$(1) \quad \mathcal{F}_n = p\mathcal{F}_{n-1} + q\mathcal{F}_{n-2}, \quad \text{with} \quad \mathcal{F}_0 = 0 \text{ and } \mathcal{F}_1 = 1,$$

which has the unique solution

$$\mathcal{F}_n = \frac{1}{\sqrt{p^2 + 4q}} (\vartheta^n - \vartheta'^n).$$

Via induction, we get

**Fact 3.** For all  $n \in \mathbb{N}_0$ , one has the identity  $\vartheta^{n+2} = \mathcal{F}_{n+2}\vartheta + q\mathcal{F}_{n+1}$ .

**Remark 1.** In the case  $p = q = 1$ , these are the well-known relations between the Fibonacci numbers  $F_n$  and the golden ratio  $\tau$ , since

$$F_n = \frac{1}{\sqrt{5}} (\tau^n - \tau'^n) \quad \text{and} \quad \tau^{n+2} = F_{n+2}\tau + F_{n+1}.$$

On the other hand, the case  $n = 0$  gives us  $\vartheta^2 = p\vartheta + q$ . Note that we also have  $\ell(w^{(n)}) = \vartheta^n$ .

In what follows, the Fourier amplitude for the geometric patch defined by  $w^{(n)}$  is denoted by  $A_n(k)$ , i.e.

$$A_n(k) = \sum_{j=1}^{|w^{(n)}|} e^{-2\pi i k x_j}$$

(note that this quantity is not normalised per point), where  $|w^{(n)}|$  is the number of letters of  $w^{(n)}$ . Following [10, Sec. 2] and using the concatenation rule, one can deduce

$$A_n(k) = f_{n-1}(k)A_{n-1}(k) + g_{n-2}(k)A_{n-2}(k).$$

Here,  $g_n$  is a sum of  $q$  exponential functions of the form  $e^{-2\pi i k \phi_0}$  and  $f_n$  is of the form  $1 + e^{-2\pi i k \vartheta^{n_0}} + \sum_{j=1}^{p-2} e^{-2\pi i k \phi_j}$ , where  $n_0 \in \{n-1, n-2\}$  and  $\phi_j \in \mathbb{R}$  for all  $j \in \{0, \dots, p-2\}$ . By [11, Thm. 3.2], the intensities can be calculated as

$$(2) \quad I(k) = \lim_{n \rightarrow \infty} \frac{|A_n(k)|^2}{\vartheta^{2n}}.$$

Any of the Pisot substitutions under investigation here, due to Hollander and Solomyak [12] and Sing [19], possesses a description as a model set, so that the constructive part of Proposition 2 applies to all  $k \in \mathbb{Q}(\vartheta)$ . In particular, one then has

$$(3) \quad \lim_{n \rightarrow \infty} \frac{A_n(k)}{\vartheta^n} = A(k) \quad \text{and} \quad I(k) = |A(k)|^2$$

with  $A(k)$  according to Theorem 1, where the construction of the underlying CPS follows from [5, 19].

To determine the limit in Eq. (2) for all remaining  $k \in \mathbb{R}$ , we need the following three Lemmata.

**Lemma 4.** [9, Cor. 2] *If  $\alpha$  is a PV number and  $\xi \notin \mathbb{Q}(\alpha)$ , one has*

$$\limsup_{n \rightarrow \infty} \{\xi \alpha^n\} - \liminf_{n \rightarrow \infty} \{\xi \alpha^n\} \geq \frac{1}{1 + \alpha}.$$

□

**Lemma 5.** [8, Thm. 1] *Let  $\alpha > 1$  be an algebraic integer and let  $\xi > 0$  be a real number. Then, the set  $\{\{\xi \alpha^n\} \mid n \in \mathbb{N}\}$  has only finitely many limit points if and only if  $\alpha$  is a PV number and  $\xi \in \mathbb{Q}(\alpha)$ .*

□

**Lemma 6.** *Let  $\alpha$  be a PV number,  $\xi \notin \mathbb{Q}(\alpha)$  and  $(y_n)_{n \in \mathbb{N}} := (\xi \alpha^n)_{n \in \mathbb{N}}$ . Then, there are numbers  $\delta = \delta(\xi, \alpha) \in (0, 1)$  and  $r = r(\xi, \alpha) \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$ :*

$$\|y_n\| < \delta \implies \|y_j\| \geq \delta \text{ for at least one } j \in \{n+1, \dots, n+r\}.$$

*Proof.* Suppose the assertion is wrong. Then, for any  $\delta \in (0, 1)$ , there is a strictly increasing sequence  $(r_m)_{m \in \mathbb{N}}$  of positive integers (hence also unbounded) and a subsequence  $(y_{n_m})_{m \in \mathbb{N}}$  of  $(y_n)_{n \in \mathbb{N}}$ , such that

$$\|y_{n_m}\| < \delta \implies \|y_j\| < \delta \text{ for all } j \in \{n_m+1, \dots, n_m+r_m\}.$$

This cannot be true for any  $\delta$  by Lemma 5, because  $(\{y_n\})_{n \in \mathbb{N}}$  has infinitely many limit points and the distance between the biggest and smallest limit point is at least  $\frac{1}{1+\alpha}$  by Lemma 4.  $\square$

Now, we can determine the intensities for wave numbers  $k \notin \mathbb{Q}(\vartheta)$  as follows.

**Theorem 7.** *For any  $k \in \mathbb{R} \setminus \mathbb{Q}(\vartheta)$ , one has  $I(k) = 0$ .*

*Proof.* Obviously, it is sufficient to show via induction that

$$\forall k \notin \mathbb{Q}(\vartheta) : \exists c = c(k) > 0, n_0 = n_0(k) \in \mathbb{N} : \forall n \geq n_0 : \frac{|A_n(k)|^2}{\vartheta^{2n}} \leq \frac{c}{n}.$$

For  $n \in \{n_0, \dots, n_0 + 2r\}$  ( $r$  will be chosen later), one just has to choose the constant  $c$  large enough. Now, let the assertion be true for a fixed  $n = n_0 + 2r$  and its predecessors  $n_0, \dots, n_0 + 2r - 1$ . Without loss of generality, let  $n_0$  be so large that

$$\frac{n_0 + 1}{n_0 - r - 1} \leq 1 + \varepsilon$$

( $\varepsilon > 0$  will be chosen later, too). By Lemma 6, there is a  $\delta \in (0, 1)$  and  $r \in \mathbb{N}$ , such that  $\|k\vartheta^j\| \geq \delta$  for at least one  $j \in \{n - r, \dots, n\}$ , i.e.

$$|1 + e^{-2\pi i k \vartheta^j}| \leq 2 - \delta'$$

for some  $\delta' \in (0, 1)$ . Thus, we have

$$|f_j| \leq p - \delta'$$

for at least one  $j \in \{n - r, \dots, n\}$  each. Without loss of generality, we can assume that  $j = n - r$ . It follows by Eq. (1) and  $|f_n| \leq p$  and  $|g_n| \leq q$

$$\begin{aligned} |A_{n+1}| &= |f_n A_n + g_{n-1} A_{n-1}| \leq p|A_n| + q|A_{n-1}| \\ &= \mathcal{F}_2|A_n| + q\mathcal{F}_1|A_{n-1}| \\ &= \mathcal{F}_2|f_{n-1} A_{n-1} + g_{n-2} A_{n-2}| + q\mathcal{F}_1|A_{n-1}| \\ &\leq \mathcal{F}_2(p|A_{n-1}| + q|A_{n-2}|) + q\mathcal{F}_1|A_{n-1}| \\ &= (p\mathcal{F}_2 + q\mathcal{F}_1)|A_{n-1}| + q\mathcal{F}_2|A_{n-2}| \\ &= \mathcal{F}_3|A_{n-1}| + q\mathcal{F}_2|A_{n-2}| \\ &\leq \dots \\ &\leq \mathcal{F}_{r+1}|A_{n-r+1}| + q\mathcal{F}_r|A_{n-r}| \\ &= \mathcal{F}_{r+1}|f_{n-r} A_{n-r} + g_{n-r-1} A_{n-r-1}| + q\mathcal{F}_r|A_{n-r}| \\ &\leq \mathcal{F}_{r+1}(|f_{n-r}| |A_{n-r}| + |g_{n-r-1}| |A_{n-r-1}|) + q\mathcal{F}_r|A_{n-r}| \\ &\leq \mathcal{F}_{r+1}((p - \delta')|A_{n-r}| + q|A_{n-r-1}|) + q\mathcal{F}_r|A_{n-r}| \\ &= ((p - \delta')\mathcal{F}_{r+1} + q\mathcal{F}_r)|A_{n-r}| + q\mathcal{F}_{r+1}|A_{n-r-1}| \\ &= (\mathcal{F}_{r+2} - \delta'')|A_{n-r}| + q\mathcal{F}_{r+1}|A_{n-r-1}| \end{aligned}$$

for  $\delta'' := \mathcal{F}_{r+1}\delta' > 0$ . Therefore, there is a  $\delta'' > 0$  and  $r \in \mathbb{N}$  such that

$$(4) \quad |A_{n+1}| \leq (\mathcal{F}_{r+2} - \delta'')|A_{n-r}| + q\mathcal{F}_{r+1}|A_{n-r-1}|.$$

By the induction hypothesis and Eq. (4), we have

$$\begin{aligned} \frac{|A_{n+1}|^2}{\vartheta^{2n+2}} &\leq \left( \frac{(\mathcal{F}_{r+2} - \delta'')|A_{n-r}| + q\mathcal{F}_{r+1}|A_{n-r-1}|}{\vartheta^{n+1}} \right)^2 \\ &\leq \left( \frac{\mathcal{F}_{r+2} - \delta''}{\vartheta^{r+1}} \cdot \sqrt{\frac{c}{n-r}} + \frac{q\mathcal{F}_{r+1}}{\vartheta^{r+2}} \cdot \sqrt{\frac{c}{n-r-1}} \right)^2. \end{aligned}$$

The right hand side is bounded by  $\frac{c}{n+1}$  if and only if

$$S := \left( \frac{\mathcal{F}_{r+2} - \delta''}{\vartheta^{r+1}} \cdot \sqrt{\frac{n+1}{n-r}} + \frac{q\mathcal{F}_{r+1}}{\vartheta^{r+2}} \cdot \sqrt{\frac{n+1}{n-r-1}} \right)^2 \leq 1.$$

This in turn is true because

$$\begin{aligned} S &\leq \vartheta^{-2r-2} \cdot \left( (\mathcal{F}_{r+2} - \delta'') \cdot \sqrt{1+\varepsilon} + \frac{q\mathcal{F}_{r+1}}{\vartheta} \cdot \sqrt{1+\varepsilon} \right)^2 \\ &= \vartheta^{-2r-2} \cdot \left( (\mathcal{F}_{r+2} - \delta'') + \frac{q\mathcal{F}_{r+1}}{\vartheta} \right)^2 \cdot (1+\varepsilon) \leq 1 \end{aligned}$$

holds if and only if

$$\varepsilon \leq \vartheta^{2r+2} \cdot \left( (\mathcal{F}_{r+2} - \delta'') + \frac{q\mathcal{F}_{r+1}}{\vartheta} \right)^{-2} - 1.$$

Now, by Fact 3,  $\varepsilon$  can be chosen positive because

$$\begin{aligned} 0 &= \vartheta^{2r+2} \cdot (\vartheta^{r+1})^{-2} - 1 \\ &= \vartheta^{2r+2} \cdot \left( \mathcal{F}_{r+2} + \frac{q\mathcal{F}_{r+1}}{\vartheta} \right)^{-2} - 1 \\ &< \vartheta^{2r+2} \cdot \left( (\mathcal{F}_{r+2} - \delta'') + \frac{q\mathcal{F}_{r+1}}{\vartheta} \right)^{-2} - 1. \end{aligned}$$

By induction, the assertion is true for all  $n \geq n_0$ . □

Let us comment on some connections with known results in the literature. There is the following link to substitution dynamical systems. Let  $\zeta$  be a primitive Pisot substitution on  $\mathcal{A} = \{a, b\}$  (for example one of the substitutions considered above) and  $w$  be a fixed point of  $\zeta$ , i.e. an element of  $\mathcal{A}^{\mathbb{Z}}$  such that  $\zeta(w) = w$ . Let  $S : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$  be the shift map defined by

$$(Sv)_k := v_{k+1},$$

and let

$$X_\zeta := \overline{\{S^j w \mid j \in \mathbb{Z}\}} \subseteq \mathcal{A}^{\mathbb{Z}}.$$

The pair  $(X_\zeta, S)$  is a topological dynamical system, called the substitution dynamical system (for  $\zeta$ ). In this situation, it is a well-known fact that  $(X_\zeta, S)$  is uniquely ergodic, i.e. there is a unique  $S$ -invariant Borel probability measure  $\mu$ . The system  $(X_\zeta, S, \mu)$  is a measure-preserving system and its spectral type is, by definition, the spectral type of the unitary operator  $U$  on  $L^2(X_\zeta, \mu)$  defined by

$$Uf(x) = f(Sx).$$

Furthermore, due to [12, Thm. 2.2], we know that  $(X_\zeta, S, \mu)$  is pure point (or has pure discrete spectrum). This means that there is a basis of  $L^2(X_\zeta, \mu)$  consisting of eigenfunctions of  $U$ . By the Halmos–von Neumann Theorem, a measure-preserving transformation is pure point if and only if it is measure-theoretically isomorphic to a translation on a compact Abelian group, see [21].

The same holds true if, instead of considering the substitution dynamical system with  $\mathbb{Z}$ -action, we have a look at the corresponding tiling dynamical system with  $\mathbb{R}$ -action; see [20] for definitions and results. This also follows from [7, Thm. 3.1].

Now, the connection between these results and diffraction theory is the following statement. Given a dynamical system on the translation bounded measures  $(\Omega, \alpha)$  with invariant probability measure  $m$ , associated unitary representation  $T_m$  by translation operators and associated diffraction measure  $\widehat{\gamma}_m$ , then  $\widehat{\gamma}_m$  is pure point if and only if  $T_m$  is pure point, see [2, Thm. 7].

The above derivation re-establishes the key result via explicit estimates of the underlying exponential sums, thus interpreting  $A(k)$  from (3) as an amplitude - despite the fact that  $\delta_\Lambda$  for the corresponding model set is not Fourier transformable as a measure. It is expected that this phenomenon is much more general, though it is presently not clear how to extend the concrete approach accordingly; see [3].

Let us now turn to consequences of Theorem 1 outside the realm of deterministic inflation rules.

#### 4. OUTLOOK

One can extend the result of Theorem 7 as follows. In [10], Godrèche and Luck introduced the concept of *random inflation tilings*, which extends the study of conventional substitutions. A mathematical rigorous treatment of a special class of such substitutions can be found in [4, 15, 16].

**Definition 1.** A substitution  $\rho : \mathcal{A}_n^* \rightarrow \mathcal{A}_n^*$  is called *stochastic* or *random* if there are  $k_1, \dots, k_n \in \mathbb{N}$  and probability vectors

$$\{\mathbf{p}_i = (p_{i1}, \dots, p_{ik_i}) \mid \mathbf{p}_i \in [0, 1]^{k_i} \text{ and } \sum_{j=1}^{k_i} p_{ij} = 1, 1 \leq i \leq n\},$$

such that

$$\rho : a_i \mapsto \begin{cases} w^{(i,1)}, & \text{with probability } p_{i1}, \\ \vdots & \vdots \\ w^{(i,k_i)}, & \text{with probability } p_{ik_i}, \end{cases}$$

for  $1 \leq i \leq n$  where each  $w^{(i,j)} \in \mathcal{A}_n^*$ . The corresponding stochastic substitution matrix is defined by

$$M_\rho := \left( \sum_{q=1}^{k_j} p_{jq} \text{card}_{a_i} w^{(j,q)} \right)_{1 \leq i, j \leq n} \in \text{Mat}(n, \mathbb{R}_{\geq 0}).$$

**Remark 2.** As in the deterministic case, a random substitution  $\rho$  is called primitive if and only if  $M_\rho$  is a primitive matrix. Note, however, that the meaning is now a stochastic one.

Now, let  $m \in \mathbb{N}$  and  $\mathbf{p}_m = (p_0, \dots, p_m)$  be a probability vector, both assumed to be fixed. The *random substitution*  $\zeta_m : \mathcal{A}_2^* \rightarrow \mathcal{A}_2^*$  is defined by

$$\zeta_m : \begin{cases} a \mapsto \begin{cases} ba^m, & \text{with probability } p_0, \\ aba^{m-1}, & \text{with probability } p_1, \\ \vdots & \vdots \\ a^{m-1}ba, & \text{with probability } p_{m-1} \\ a^mb, & \text{with probability } p_m, \end{cases} \\ b \mapsto a, \end{cases}$$

and the one-parameter family  $\mathcal{R} = \{\zeta_m\}_{m \in \mathbb{N}}$  is called the family of *random noble means substitutions* (RNMS). The stochastic substitution matrix is given by

$$M_m := M_{\zeta_m} = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix},$$

which is independent of the probability vector  $\mathbf{p}_m$ . The eigenvalues are  $\lambda_m := \frac{m + \sqrt{m^2 + 4}}{2}$  and  $\lambda'_m := \frac{m - \sqrt{m^2 + 4}}{2}$ , while the left eigenvector is  $(\lambda_m, 1)$ . Now, one can prove (almost) along the same lines as in Theorem 7 that  $I(k) = 0$  for all  $k \notin \mathbb{Q}(\lambda_m)$ . Even more, with a modification of Lemma 6, one can prove the following result.

**Proposition 8.** *Let  $m \in \mathbb{N}$  and consider the RNMS  $\zeta_m$ . For any wave number  $k \in \mathbb{R} \setminus \frac{\mathbb{Z}[\lambda_m]}{\sqrt{m^2 + 4}}$  (i.e. for any  $k$  that is not in the Fourier module), we have  $I(k) = 0$ .*

*Sketch of proof.* For  $k \notin \mathbb{Q}(\lambda_m)$ , one can argue as in the proof of Theorem 7. The functions  $f_n$  and  $g_n$  are again sums of exponential functions as above (maybe multiplied by some  $p_i$ ) and are again bounded by  $m$  respectively 1.

For  $k \in \mathbb{Q}(\lambda_m) \setminus \frac{\mathbb{Z}[\lambda_m]}{\sqrt{m^2 + 4}}$ , the assertion of Lemma 6 is obviously still true and one can argue again as in the proof of Theorem 7.  $\square$

**Remark 3.** In a deterministic setting, substitution dynamical systems are rather well understood; see e.g. [1, 18]. Far less is known in the realm of systems inducing mixed spectra. In this case, the understanding in the presence of entropy is only at its beginning, and it is desirable to work out particular examples like the RNMS. For more information about the RNMS, see [4, 15, 16].

As before, various generalisations should be possible, in particular in view of the fact that Godrèche and Luck [10] also treat planar analogues. At present, it is not clear though how the above approach can be extended to cover planar systems.

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